Asymptotic parabolicity for strongly damped wave equations

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This paper is dedicated to Fritz Gesztesy on his 60th birthday .

Abstract. For S a positive selfadjoint operator on a Hilbert space,

$$\frac{d^2u}{dt}(t) + 2F(S)\frac{du}{dt}(t) + S^2u(t) = 0$$

describes a class of wave equations with strong friction or damping if F is a positive Borel function. Under suitable hypotheses, it is shown that

$$u(t) = v(t) + w(t)$$

where v satisfies

$$2F(S)\frac{dv}{dt}(t) + S^2v(t) = 0$$

and

$$\frac{w(t)}{\|v(t)\|} \to 0, \text{ as } t \to +\infty.$$

The required initial condition $v(0)$ is given in a canonical way in terms of $u(0)$, $u'(0)$.

1. Introduction

Let S be an injective nonnegative selfadjoint operator on a complex Hilbert space $\mathcal{H}$. That is $S = S^* \geq 0$, $0 \notin \sigma_p(S)$. Consider the damped wave equation

$$u''(t) + 2Bu'(t) + S^2u(t) = 0, \quad t \geq 0, \tag{1.1}$$

with initial conditions

$$u(0) = f, \quad u'(0) = g; \tag{1.2}$$

here $' = d/dt$. When $B = 0$, (1.1) reduces to the wave equation, and the corresponding heat equation normally considered is

$$v'(t) + S^2v(t) = 0.$$

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In this paper we take B to be a positive selfadjoint operator which commutes with S and is "smaller than S". More precisely, we assume that

$$0 = \inf \sigma(S), \tag{1.3}$$

i.e., 0 is the spectrum of $S = S^* \geq 0$, but is not an eigenvalue, F is a continuous function from $(0, +\infty)$ to $(0, +\infty)$, $B = F(S)$, and F satisfies: there exists $\gamma > 0$ such that

$$\begin{cases} F(x) > x & \text{for } 0 < x < \gamma, \\ F(\gamma) = \gamma, & \\ F(x) < x & \text{for } x > \gamma, \\ \limsup_{x \to 0^+} F(x) < +\infty, & \\ \liminf_{x \to +\infty} ((1-\delta)x - F(x)) \geq 0, & \text{for some } \delta > 0. \end{cases} \tag{1.4}$$

We make a further comment on (1.3). Think of S^2 as $-\Delta$ with suitable boundary conditions, acting on $H = L^2(\Omega)$ where Ω is a domain in $\mathbb{R}^N$. Then (1.3) implies that S cannot have compact resolvent, and we are thus led to work exclusively in unbounded domains Ω.

The operator B represents a general friction coefficient. The most common case is the telegraph equation in which case

$$B = aI$$

where a is a positive constant. In this case (1.4) holds with $\gamma = a$. Another simple case is

$$B = aS^\alpha$$

where the constants a, α satisfy

$$a > 0, \quad \alpha \in [0, 1).$$

In this case, $\gamma = a^{\frac{1}{1-\alpha}}$ in (1.4). The only interesting case is when S is unbounded. The *strongly damped wave equation* refers to the case when B is also unbounded.

Our main result, Theorem 3.1, can be stated informally as follows. Let $S, B = F(S)$ be as above and suppose f, g are such that (1.1), (1.2) has a unique solution u. Consider the corresponding first order (in t) equation, obtained by erasing $u''(t)$ in (1.1) and replacing u by v:

$$2Bv'(t) + S^2 v(t) = 0, \quad t \geq 0, \tag{1.5}$$

with initial condition

$$v(0) = h. \tag{1.6}$$

This vector h is given by

$$h = \frac{1}{2} \chi_{(0,\gamma)}(S) \{ (F(S)^2 - S^2)^{\frac{1}{2}} (F(S) f + g) + f \},$$

a formula which will be derived and explained later; and our conclusion requires that f, g are such that $h \neq 0$. We will show that

$$l(t) := \frac{\|u(t) - v(t)\|}{\|v(t)\|} \to 0, \tag{1.7}$$

as $t \to 0$, and we will find closed subspaces $\mathcal{E}_n$ of $\mathcal{H}$ such that $\mathcal{E}_n \subset \mathcal{E}_{n+1}$,

$$\bigcup_{n=1}^{\infty} \mathcal{E}_n \text{ is dense in } \mathcal{H},$$

and

$$l(t) \leq C_n e^{-\epsilon_n t}$$

for $f, g \in \mathcal{E}_n$ where C_n, ϵ_n are positive constants. In general, there is no rate of convergence in (1.2) that works for all solutions.

The point of the theorem is that, for large times the solution of the "hyperbolic equation" (1.1) looks like the solution of the "parabolic equation" (1.5). In the telegraph equation case when $B = F(S) = aI$, (1.5) becomes

$$2av'(t) + S^2 v(t) = 0,$$

which, with (1.6), is solved by

$$v(t) = e^{-\frac{t}{2a}S^2} h.$$

In the case of strong damping, the solution of the limiting parabolic problem is

$$v(t) = e^{-\frac{t}{2}B^{-1}S^2} h.$$

Think of $S = (-\Delta)^{\frac{1}{2}}$ on $L^2(\mathbb{R}^N)$ and

$$B = aS^\alpha = a(-\Delta)^{\frac{\alpha}{2}},$$

$0 < \alpha < 1$. Then $B^{-1}S^2 = \dfrac{1}{a}(-\Delta)^{1-\frac{\alpha}{2}}$ with domain

$$D(B^{-1}S^2) = H^{2-\alpha}(\mathbb{R}^N),$$

while

$$D(S^2) = H^2(\mathbb{R}^N);$$

here we use the standard Sobolev space notation. Thus $B^{-1}S^2$ is a pseudodifferential operator of lower order $2 - \alpha$ than that of the Laplacian, unless $\alpha = 0$ in which case we have the telegraph equation.

It has long been "known" that the telegraph equation is an asymptotic approximant for the heat equation, especially in the case of $S^2 = -d^2/dx^2$ on $L^2(\mathbb{R})$. The pioneers in this area include G. I. Taylor [**12**] in 1922, S. Goldstein [**8**] in 1938 and M. Kac [**9**] in 1956.

Some of the associated random walk ideas are discussed in [**4**], which eventually led to [**1**], the main theorem in which is the special case of our main theorem here with $B = aI$. The importance of the interesting case of strong damping was recognized by Fritz Gesztesy and is discussed in detail in [**5**]. For additional results on strong damping, see [**7**].

Section 2 reviews some spectral theory. Section 3 contains the statement and the proof of our main result. The proof is patterned after that in [**1**], but it in fact is simplified and streamlined. Section 4 contains examples.

2. Selfadjoint and normal operators

Let S be a selfadjoint operator on $\mathcal{H}$ with spectrum $\sigma(S)$. By the spectral Theorem there exists an L^2 space $L^2(\Lambda, \Sigma, \nu)$ and a unitary operator

$$W : \mathcal{H} \to L^2(\Lambda, \Sigma, \nu)$$

such that S is unitarily equivalent, via W, to the maximally defined operator of multiplication by a $\Sigma-$measurable function

$$m : \Lambda \to \sigma(S) \subset \mathbb{R},$$

i.e.,

$$Sf = W^{-1} M_m W f,$$

for

$$f \in D(S) = \{W^{-1} g \in \mathcal{H} : \ mg \in L^2(\Lambda, \Sigma, \nu)\}$$

and

$$(M_m g)(x) = m(x) g(x), \quad \text{for } x \in \Lambda, g \in L^2(\Lambda, \Sigma, \nu).$$

Two selfadjoint operators S_1, S_2 *commute* if and only if the bounded operators

$$(\lambda_1 I - S_1)^{-1}, (\lambda_2 I - S_2)^{-1}$$

commute for all $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ if and only if

$$e^{itS_1} \quad \text{and} \quad e^{isS_2}$$

commute for all $t, s \in \mathbb{R}$. Similarly, two normal operators N_1, N_2 with

$$\sup \operatorname{Re} \sigma(N_j) < +\infty, \quad j = 1, 2$$

commute if and only if e^{tN_1} and e^{sN_2} commute for all $t, s \geq 0$; here N is normal means $N = S_1 + iS_2$ where S_1, S_2 are commuting selfadjoint operators.

The functional calculus for S selfadjoint says that for every Borel measurable function F from $\sigma(S) \subset \mathbb{R}$ to $\mathbb{C}$, $F(S)$ defined by

$$F(S) = W^{-1} M_{F(m)} W$$

is normal, and any two of these operators commute. Moreover,

$$F \to F(S)$$

is linear and is an algebra homomorphism, thus

$$F_1(S) F_2(S) = (F_1 F_2)(S),$$

etc. Also, $F(S)$ is bounded on $\mathcal{H}$ if and only if F is bounded on $\sigma(S)$, and $F(S)$ is selfadjoint if and only if F is real valued. And for $S = S^*$, $F(S)$ is semibounded (above or below) if and only if $F(\sigma(S))$ is, in $\mathbb{R}$.

In particular, for Γ a Borel set in $[0, +\infty)$, $P_\Gamma = \chi_\Gamma(S)$ is the orthogonal projection of $\mathcal{H}$ onto $\chi_\Gamma(S)(\mathcal{H})$; and

$$P_\Gamma F(S) = F(S) P_\Gamma = P_\Gamma F(S P_\Gamma)$$

is the part of $F(S)$ in Γ, and its spectrum is contained in Γ.

If F_1, F_2 are complex Borel functions on $\sigma(S)$ that are bounded above, it follows that $F_j(S)$ and $\Sigma_{k=1}^n F_k(S)$ generate (C_0) semigroups on $\mathcal{H}$ and

$$e^{t \Sigma_{k=1}^n F_k(S)} = \Pi_{k=1}^n e^{t F_k(S)}, \tag{2.1}$$

and the product can be taken in any order. Finally, if $L = F(S) = L^* \geq 0$, then $[L]^{\frac{1}{2}}$ denotes the unique nonnegative square root of L.

For more on the spectral theorem, see the books [**6**], [**10**], [**11**].

3. The main result

Consider the problem (1.1), (1.2), which we rewrite as

$$u'' + 2Bu' + S^2u = 0, \quad t \geq 0, \tag{3.1}$$

$$u(0) = f, u'(0) = g, \tag{3.2}$$

where $S = S^* \geq 0$ on $\mathcal{H}$,

$$\inf \sigma(S) = 0 \notin \sigma_p(S), \tag{3.3}$$

$B = F(S)$ where F is a continuous function from $(0, +\infty)$ to $(0, +\infty)$ which is bounded near zero and strictly less than the identity function near infinity, in the sense that for some $\delta > 0$,

$$\liminf_{x\to+\infty}((1-\delta)x - F(x)) \geq 0. \tag{3.4}$$

We also assume there exists $\gamma > 0$ such that (1.4) holds, namely

$$\begin{cases} F(x) > x & \text{for } 0 < x < \gamma, \\ F(\gamma) = \gamma, \\ F(x) < x & \text{for } x > \gamma, \\ \limsup_{x\to 0^+} F(x) < +\infty. \end{cases} \tag{3.5}$$

Let Γ be the open interval $(0, \gamma)$ and let

$$P_\Gamma = \chi_\Gamma(S). \tag{3.6}$$

Theorem 3.1. *Assume all the statements in the above paragraph. Let v be the solution of the corresponding "parabolic" equation*

$$2Bv' + S^2v = 0, \tag{3.7}$$

obtained by deleting the second derivative term in (3.1). *Let*

$$v(0) = h := \frac{1}{2}P_\Gamma(f + [(B^2 - S^2)P_\Gamma]^{\frac{1}{2}}(Bf + g)). \tag{3.8}$$

Then, for u the solution of (3.1), (3.2),

$$u(t) = v(t)(1 + o(t)) \tag{3.9}$$

holds as $t \to +\infty$, provided $h \neq 0$. Moreover, if $\Gamma_n = \left[\frac{1}{n}, \delta - \frac{1}{n}\right]$ and if $0 \neq h \in P_{\Gamma_n}(\mathcal{H})$ for some $n \in \mathbb{N}$, then

$$u(t) = v(t)(1 + o(e^{-\epsilon_n t})) \tag{3.10}$$

for some $\epsilon_n > 0$.

Remark 3.2. Note that

$$P_{\Gamma_n}(\mathcal{H}) \subset P_{\Gamma_{n+1}}(\mathcal{H}),$$

and $\bigcup_{n=1}^\infty P_{\Gamma_n}(\mathcal{H})$ is dense in $P_\Gamma(\mathcal{H})$.

Proof. First recall that the square root of $(B^2 - S^2)P_\Gamma$ refers to the non-negative square root. We first show that the problem (3.1), (3.2) is wellposed, by showing that it is governed by a (C_0) contraction semigroup.

We first treat the case of $B = 0$ in (3.1).

Rewrite (3.1), (3.2) as, for $U = \begin{pmatrix} Su \\ u' \end{pmatrix}$,

$$\begin{aligned} U' &= \begin{pmatrix} Su' \\ u'' \end{pmatrix} = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \begin{pmatrix} Su \\ u' \end{pmatrix} = GU, \\ U(0) &= L = \begin{pmatrix} Sf \\ g \end{pmatrix}. \end{aligned} \tag{3.11}$$

Let $\mathcal{K}$ be the completion of $D(S) \oplus D(S)$ in the norm

$$\left\| \begin{pmatrix} m \\ n \end{pmatrix} \right\|_{\mathcal{K}} = (\|m\|^2 + \|n\|^2)^{\frac{1}{2}},$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$. Then G defined by (3.11) is skewadjoint on $\mathcal{K}$ and thus generates a (C_0) unitary group by Stone's Theorem (cf. [**6**], [**10**], [**11**]). We consider this as a semigroup since we are only concerned with times $t \geq 0$. Next, (3.1) can be written in $\mathcal{K}$ as

$$U' = (G + P)U$$

where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & -2B \end{pmatrix}.$$

Since $B = B^* \geq 0$,

$$\begin{aligned} \left\| P \begin{pmatrix} m \\ n \end{pmatrix} \right\|_{\mathcal{K}} &= \|2Bn\| \leq (1-\epsilon)\|Sn\| + C_\epsilon \|n\| \\ &\leq (1-\epsilon) \left\| P \begin{pmatrix} m \\ n \end{pmatrix} \right\|_{\mathcal{K}} + C_\epsilon \left\| \begin{pmatrix} m \\ n \end{pmatrix} \right\|_{\mathcal{K}} \end{aligned}$$

for some $\epsilon > 0$ and a corresponding $C_\epsilon > 0$, for all $n \in D(S) \subset D(B)$, thanks to (3.4) and the last line in (3.5) (or (1.4)). Namely, write $B = B_1 + B_2 := BP_{(0,M)} + BP_{[M,\infty)}$, where M is such that

$$x \geq F(x) + \delta x,$$

i.e.

$$F(x) \leq (1-\delta)x$$

for $x \geq M$ and $F(x)$ is bounded in $[0, M]$. Thus B_1 is bounded, $B_2 = B_2^* \geq 0$ and

$$\|B_2 n\| \leq (1-\delta)\|Sn\| + \|B_2\|\|n\|$$

for all $n \in D(S)$. Thus for $N = \begin{pmatrix} m \\ n \end{pmatrix}$,

$$\|PN\|_{\mathcal{K}} \leq (1-\delta)\|GN\|_{\mathcal{K}} + M\|N\|_{\mathcal{K}}$$

where $\delta > 0$ and $M = \|B_2\|$. It follows (cf. e.g. [**6**], [**10**], [**11**]) that $G + P$ is $m-$dissipative and generates a (C_0) contraction semigroup on $\mathcal{K}$, since P is obviously dissipative. Then (3.1), (3.2) has a unique strongly C^2 solution (resp. mild solution) if $f \in D(S^2), g \in D(S)$ (resp. $f \in D(S), g \in \mathcal{H}$): cf. [**6**, Chapter 2, Theorem 7.8].

We shall express the unique solution using d'Alembert's formula. We seek solution of the form

$$u(t) = e^{tC}h$$

where C is a Borel function of S. By (3.1), C must satisfy

$$C^2 + 2BC + S^2 = 0.$$

Formally,

$$C = C_\pm = -B \pm (B^2 - S^2)^{\frac{1}{2}}.$$

Selfadjoint operators have many square roots, but nonnegative selfadjoint operators have unique nonnegative square roots. Thus we uniquely define $C_\pm$ by

$$C_\pm = -B \pm (Q_0 + iQ) \tag{3.12}$$

where

$$Q_0 = [(B^2 - S^2)\chi_{(0,\gamma)}(S)]^{\frac{1}{2}}, Q = [(S^2 - B^2)|\chi_{[\gamma,+\infty)}(S)]^{\frac{1}{2}}. \tag{3.13}$$

Thus the solution u of (3.1), (3.2) can be written as

$$u(t) = e^{tC_+}h_+ + e^{tC_-}h_-,$$

where $C_\pm$ are defined by (3.12), (3.13). There are strong C^2 solutions (resp. mild solutions) if and only if $h_\pm \in D(S^2)$ (resp. $h_\pm \in D(S)$).

Given $f = u(0), g = u'(0)$, we obtain $h_\pm$ by inverting the 2×2 system

$$\begin{aligned} f &= h_+ + h_- \\ g &= C_+h_+ + C_-h_-. \end{aligned}$$

An elementary calculation gives

$$h_- = \frac{1}{2}(f - (Q_0 + iQ)^{-1}(Bf + g)) \tag{3.14}$$

$$h_+ = \frac{1}{2}(f + (Q_0 + iQ)^{-1}(Bf + g)). \tag{3.15}$$

Write

$$u = u_1 + u_2 + u_3$$

where

$$\begin{aligned} u_1(t) &= e^{tC_+}P_{(0,\gamma)}h_+, \\ u_2(t) &= e^{tC_+}P_{[\gamma,+\infty)}h_+, \\ u_3(t) &= e^{tC_-}h_-. \end{aligned}$$

First,

$$\|u_3(t)\| = \|e^{-itQ}e^{-tQ_0}e^{-tB}h_-\| \leq \|e^{-tB}h_-\|$$

since e^{-itQ} is unitary and $\|e^{-tQ_0}\| \leq 1$. Next,

$$\|u_2(t)\| = \|e^{itQ}e^{-tB}P_{[\gamma,+\infty)}h_+\| \leq \|e^{-tB}h_+\|.$$

The next estimate is the key one.

For

$$h := P_{(0,\gamma)}(h_+), \tag{3.16}$$

$$\|u_1(t)\| = \|e^{t(-B+Q_0)}h_+\|.$$

We know that $h \in P_{(0,\gamma)}(\mathcal{H})$: assume

$$0 \neq h \in P_{[\delta,\gamma-\delta]}(\mathcal{H}) =: \mathcal{H}_\delta \tag{3.17}$$

for some $\delta > 0$. Let

$$Q_{0\delta} = Q_0 P_{[\delta,\gamma-\delta]}.$$

Then, since $F(x) > x$ on $[\delta, \gamma - \delta]$, $F(x) - x \geq \epsilon$ on $[\delta, \gamma - \delta]$ for some $\epsilon > 0$. Thus

$$Q_{0\delta} \geq \epsilon I.$$

Consequently

$$\|u_1(t)\| \geq e^{\epsilon t}\|e^{-tB}h\|.$$

It follows that for some constant C_0,

$$\|u_2(t)\| + \|u_3(t)\| \leq C_0 e^{-\epsilon t}\|u_1(t)\|.$$

Thus

$$u(t) = u_1(t)(1 + O(e^{-\epsilon t})). \tag{3.18}$$

We must show that this holds with u_1 replaced by v.

The unique solution of (3.7) is

$$v(t) = e^{-\frac{t}{2}B^{-1}S^2}h.$$

Note that h as defined by (3.8) is $P_\Gamma h_+$ where h_+ is as in (3.15). To compare u_1 with v, we need Taylor's formula with integral remainder, which for $g \in C^3[0,l]$ and for some $l > 0$ says that

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{1}{2}\int_0^x (x-y)^2 g'''(y)dy.$$

Applying this to

$$g(x) = 1 - (1-x)^{\frac{1}{2}}, \quad 0 < x < 1, \tag{3.19}$$

yields

$$g(L)f = \frac{1}{2}Lf + \frac{1}{8}L^2 f + Rf \tag{3.20}$$

where R is a bounded operator commuting with L and satisfying

$$R = R^* \geq 0.$$

Consequently

$$\begin{aligned} \|u_1(t) - v(t)\| &= \|e^{-tB(-I+B^{-1}Q_0)}h - e^{-\frac{t}{2}B^{-1}S^2}h\| \\ &= \|e^{-tB[I-(B^{-2}Q_0^2)^{\frac{1}{2}}]}h - e^{-\frac{t}{2}B^{-1}S^2}h\| \\ &= \|e^{-\frac{t}{2}B^{-1}S^2}\{e^{-\frac{t}{8}L^2}e^{-tR} - I\}h\| \end{aligned} \tag{3.21}$$

by (3.19), (3.20) with $L = B^{-2}(B^2 - S^2)P_{(0,\gamma)} = (I - B^{-2}S^{-2})P_{(0,\gamma)}$.

We have

$$\zeta_1 I \leq R \leq \zeta_2 I$$

on $\mathcal{H}_\delta$ for some constants $0 < \zeta_1 < \zeta_2 < +\infty$. Furthermore, we also have

$$\zeta_3 I \leq L \leq \zeta_4 I$$

on $\mathcal{H}_\delta$ for some positive constants ζ_3, ζ_4. It now follows from (3.20) that

$$\|u_1(t) - v(t)\| = \|e^{-\frac{t}{2}B^{-1}S^2}(I - e^{-\frac{t}{8}L}e^{-tR}h)\|$$

and

$$\|e^{-\frac{t}{8}L}e^{-tR}h\| \leq e^{-t\zeta_5}\|h\|$$

where

$$\zeta_5 = \frac{\zeta_3}{8} + \zeta_1 > 0.$$

Consequently

$$\|u(t) - v(t)\| \leq \|v(t)\| O(e^{-t\zeta_5}).$$

Combining this inequality with (3.18) yields the desired asymptotic relation

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} = o(e^{-t\epsilon_\delta})$$

for some $\epsilon_\delta > 0$.

Now let $0 \neq h \in P_{(0,\gamma)}(\mathcal{H})$. We must show that

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} \to 0, \quad \text{and } t \to +\infty. \tag{3.22}$$

We proceed by contradiction. Suppose (3.22) fails to hold for some $h \neq 0$ in $P_{(0,\gamma)}(\mathcal{H})$. Then, there exists $\epsilon_1 > 0$ and $t_n \to +\infty$ such that

$$\frac{\|u(t_n) - v(t_n)\|}{\|v(t_n)\|} \geq \epsilon_1 \tag{3.23}$$

for all $n \in \mathbb{N}$. Choose $\delta > 0$ and $\tilde{h} \in \mathcal{H}_\delta = P_{[\delta,\gamma-\delta]}(\mathcal{H})$ (depending on ϵ_1) such that

$$\|h - \tilde{h}\| < \frac{\epsilon_1}{4}$$

and let $\tilde{f}$, $\tilde{g}$ be the corresponding initial data. Note that

$$P_{[\gamma,+\infty)} l = P_{[\gamma,+\infty)} \tilde{l}$$

for $l = f, g$, and f and g are modified only on the subspace $P_\Lambda(\mathcal{H})$

$$\Lambda := [\delta - \delta_1, \delta + \delta_1] \cup [\gamma - \delta - \delta_1, \gamma - \delta + \delta_1],$$

for some $\delta_1 > 0$ which can be chosen to be arbitrarily small. In particular, given $\epsilon_2 > 0$ we may choose $\tilde{f}, \tilde{g}$ as above and additionally satisfying

$$\|f - \tilde{f}\| + \|g - \tilde{g}\| < \epsilon_2,$$

$$\frac{\|\tilde{h}\|}{\|h\|} \in [1 - \epsilon_2, 1 + \epsilon_2].$$

It follows that

$$\|u(t) - \tilde{u}(t)\|,\ \|v(t) - \tilde{v}(t)\| < \frac{\epsilon_1}{4}$$

for all $t > 0$. Consequently

$$\begin{aligned} \frac{\|u(t) - v(t)\|}{\|v(t)\|} &\leq \frac{\|\tilde{u}(t) - \tilde{v}(t)\|}{\|\tilde{v}(t)\|} \left(\frac{1 + \epsilon_2}{1 - \epsilon_2} \right) + \frac{\epsilon_1}{4} \\ &\leq \tau_0 e^{-\epsilon_3 t} \left(\frac{1 + \epsilon_2}{1 - \epsilon_2} \right) + \frac{\epsilon_1}{4} \to \frac{\epsilon_1}{4}, \end{aligned} \tag{3.24}$$

as $t \to +\infty$, since $0 \neq \tilde{h} \in \mathcal{H}_\delta$, and τ_0, ϵ_3 are positive constants depending on δ. But (3.24) contradicts (3.23) for $t = t_n$ with n large enough. It follows that (3.22) holds. This completes the proof of Theorem 3.1. □

We remark that, in general, there does not exist a rate of convergence in (3.22) which works for all $0 \neq h \in \mathcal{H}_\gamma$. This follows from a careful examination of [**1**] and [**2**, Section 3].

4. Examples

Example 4.1. This is the simplest example. Take

$$S^2 := -\Delta + w^2 I \quad \text{on } L^2(\mathbb{R}^N)$$

for $w > 0$. Similarly we can define

$$S_k^2 := S^{2k} = (-\Delta + w^2 I)^k \quad \text{on } L^2(\mathbb{R}^N),\ k \in \mathbb{N}.$$

Theorem 3.1 applies (with $k \in \mathbb{N}$, $w > 0$ fixed) if we take

$$B = aS_k^\alpha = aS^{\alpha k}$$

for $a > 0$, $\alpha \in [0, 1)$. The resulting damped wave equation is

$$u_{tt} + 2a(-\Delta + w^2)^{\frac{\alpha k}{2}} u_t + (-\Delta + w^2)^k u = 0. \tag{4.1}$$

This is a pseudo differential equation unless $\frac{\alpha k}{2} \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, in which case it is a partial differential equation. The simplest case of this is $k = 3$, $\alpha = \frac{2}{3}$, in which case (4.1) reduces to

$$u_{tt} + 2a(-\Delta u_t + w^2 u_t) + (-\Delta + w^2)^3 u = 0.$$

As noted earlier, γ is given by

$$\gamma = a^{\frac{1}{1-\alpha}}$$

in the case of (4.1), for all $k \in \mathbb{N}$.

Example 4.2. We start with a brief summary of the "Wentzell Laplacian" discussed in [**2**] (cf. also [**3**]). Let Ω be an unbounded domain in $\mathbb{R}^N$ with nonempty boundary $\partial\Omega$, such that for every $R > 0$ there exists a ball $B(x_R, R)$ in Ω. Let $A(x)$ be an $N \times N$ matrix for $x \in \overline{\Omega}$ such that $A(x)$ is real, Hermitian and

$$\alpha_1 |\xi|^2 \le A(x)\xi \cdot \xi \le \alpha_2 |\xi|^2 \tag{4.2}$$

for all $x \in \overline{\Omega}, \xi \in \mathbb{R}^N$, where

$$0 < \alpha_1 \le \alpha_2 < \infty$$

are constants. Similarly let $B(x)$ for $x \in \partial\Omega$ be a real Hermitian $(N-1)\times(N-1)$ matrix satisfying (4.2) with the same α_1, α_2. Assume $\partial\Omega, A, B$, and γ, β (introduced below) are sufficiently smooth. Define distributional differential operators on Ω (resp. $\partial\Omega$) by

$$Lu_1 = \nabla \cdot (A(x)\nabla u_1),$$
$$L_\partial u_2 = \nabla_\tau \cdot (B(x)\nabla_\tau u_2)$$

for u_1 (resp. u_2) defined on Ω (resp. $\partial\Omega$). Here ∇_τ is the tangential gradient on $\partial\Omega$. The wave equation (without damping) we consider is

$$u_{tt} = Lu \quad \text{in } \Omega, \tag{4.3}$$

$$Lu + \beta \partial_\nu^A u + \gamma u - q\beta L_\partial u = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

Here the conormal derivative term is

$$\partial_\nu^A u = (A\nabla u) \cdot \nu$$

at $x \in \partial\Omega$, where ν is the unit outer normal to $\partial\Omega$ at x; $\beta, \gamma \in C^1(\partial\Omega, \mathbb{R}), \beta > 0, \gamma \ge 0$, $\beta, \frac{1}{\beta}$ and γ are bounded, and $q \in [0, +\infty)$.

In [**2**], it is explained how the problem (4.3), (4.4) can be rewritten as

$$u'' + S^2 u = 0$$

$$u(0) = f, \quad u'(0) = g$$

where the Hilbert space is

$$\mathcal{H} = L^2(\Omega, dx) \oplus L^2\left(\partial\Omega, \frac{d\Gamma}{\beta}\right).$$

Vectors in $\mathcal{H}$ are represented by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $u_1 \in L^2(\Omega, dx)$ and $u_2 \in L^2\left(\partial\Omega, \frac{d\Gamma}{\beta}\right)$. The norm in $\mathcal{H}$ is given by

$$\|u\|_{\mathcal{H}} = \{\|u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2\left(\partial\Omega, \frac{d\Gamma}{\beta}\right)}\}^{\frac{1}{2}}.$$

Here we write $d\Gamma$ rather than the usual dS for the element of surface measure, since the letter S already is being used to denote the basic operator.

The operator S^2 has the matrix representation

$$S^2 = \begin{pmatrix} -L & 0 \\ \beta\partial_\nu^A & \gamma - q\beta L_\partial \end{pmatrix}.$$

In [**2**] it was shown that $S = [S^2]^{\frac{1}{2}}$, with a suitable domain, satisfies

$$S = S^* \geq 0, \; 0 = \inf \sigma(S), \; 0 \notin \sigma_p(S).$$

Furthermore, for all $u \in D(S)$, we have $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_2 = \operatorname{tr}(u_1)$, the trace of u_1. As in Example 4.1, Theorem 3.1 applies to

$$u'' + 2aS^{\frac{\alpha k}{2}}u' + S^{2k}u = 0, \tag{4.5}$$

$$u(0) = f, \; u'(0) = g, \quad k \in \mathbb{N}.$$

Again, this is a partial differential equation only when $\frac{\alpha k}{2} \in \mathbb{N}$. If $k = 3$ and $\alpha = \frac{2}{3}$, the corresponding parabolic problem is

$$v' + \frac{1}{2a}S^4 v = 0, \quad v(0) = h.$$

The boundary conditions associated with (4.5) are

$$Lw + \beta\partial_\nu^A w + \gamma w - qL_\partial w = 0 \quad \text{on } \partial\Omega$$

for $w = S^{2j}u$, $j = 0, 1, ..., k-1$.

Example 4.3 The simplest example of unidirectional waves in one dimension are described by the equation (for $t, x \in \mathbb{R}$)

$$u_t = cu_x + bu_{xxx} =: Mu, \tag{4.6}$$

where $(b, c) \in \mathbb{R}^2 \setminus \{(0,0)\}$. The most common case is $c \neq 0, b = 0$, in which case the corresponding equation for bidirectional waves is

$$\left(\frac{\partial}{\partial t} - M\right)\left(\frac{\partial}{\partial t} + M\right)u = u_{tt} - c^2 u_{xx} = 0.$$

The case of $b \neq 0$ is the Airy equation, and (4.6) is the linearization of the KdV equation

$$u_t = cu_x + bu_{xxx} + c_1 u u_x.$$

For $c = 0 \neq b$, the bidirectional version of (4.6) is

$$\left(\frac{\partial}{\partial t} - M\right)\left(\frac{\partial}{\partial t} + M\right)u = u_{tt} - b^2 u_{xxxxxx} = 0.$$

Now, let $\mathcal{H} = L^2(\mathbb{R})$, $D = \frac{d}{dx}$ and $T = -D^2 = T^* \geq 0$. Let

$$S^2 = T^3 + a_0T^2 + a_1T,$$

where $a_0,\, a_1 \in [0, +\infty)$. Consider

$$u_{tt} - 2au_{xxt} - u_{xxxxxx} + a_0u_{xxxx} - a_1u_{xx} = 0,$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

In this case,

$$B = aT = F(S) = F(T^3 + a_0T^2 + a_1T).$$

For $x > 0$, we want to consider the function

$$G(x) = \frac{1}{a}(x^3 + a_0x^2 + a_1x),$$

so that $G(x) - x$ is negative in $(0, \gamma)$ and positive on (γ, ∞) for some $\gamma > 0$. But

$$\frac{d}{dx}\left(\frac{G(x) - x}{x}\right) = \frac{d}{dx}\left(\frac{1}{a}(x^2 + a_0x + (a_1 - a))\right) = \frac{1}{a}(2x + a_0) > 0,$$

and $G(x) = x$ for $x \neq 0$ if and only if

$$x = \frac{1}{2}(-a_0 \pm \sqrt{a_0^2 - 4(a_1 - a)}\ ).$$

Thus we get exactly one positive root if and only if

$$a > a_1 + \frac{a_0^2}{4}, \tag{4.7}$$

which we assume. It is now elementary to check that $B = F(S)$ and F satisfies the assumptions of Theorem 3.1. In this case

$$\gamma = \frac{1}{2}(-a_0 + \sqrt{a_0^2 - 4(a_1 - a)}\ ).$$

References

[1] T. Clarke, E. C. Eckstein, J. A. Goldstein, *Asymptotic analysis of the abstract telegraph equation*, Diff. Int. Eqns. **21** (2008), 433–442.

[2] T. Clarke, G. R. Goldstein, J. A. Goldstein, S. Romanelli, *The Wentzell telegraph equation: asymptotics and continuous dependence on the boundary conditions*, Commun. Appl. Math. **15**, (2011), 313–324.

[3] G. M. Coclite, A. Favini, C. G. Gal, G. R. Goldstein, J. A. Goldstein, E. Obrecht, S. Romanelli, *The role of Wentzell boundary conditions in linear and nonlinear analysis.* In: Advances in Nonlinear Analysis: Theory, Methods and Applications, Cambridge, S. Sivasundaram, ed., 2009, 270–291.

[4] E.C. Eckstein, J. A. Goldstein, M. Leggas, *The mathematics of suspensions: Kac walks and asymptotic analyticity.* In: Proceedings of the Fourth Mississippi State Conference on Differential Equations and Computational Simulations (1999), Vol. **3** Electron. J. Differ. Equ. Conf., San Marcos, TX, 2000, Southwest Texas State Univ., 39–50.

[5] F. Gesztesy, J. A. Goldstein, H. Holder, G. Teschl, *Abstract wave equations and associated Dirac-type operators*, Ann. Mat. Pura Appl. **191** (2012), 631-676.

[6] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, New York, 1985.

[7] J. A. Goldstein, G. Reyes, *Asymptotic equipartition of operator weighted energy in damped wave equations*, Asymptotic Anal. (2012), 17 pp, to appear.

[8] S. Goldstein, *On diffusion by discontinuous movements, and on the telegraph equation*, Quart. J. Math. Mech. **4** (1951), 129–156.

[9] M. Kac, *A stochastic model related to the telegrapher's equation.* In: Some stochastic problems in physics and mathematics. Magnolia Petroleum Co., Lectures in Pure and Applied Science, No. 2, 1956. Reprinted in: Rocky Mtn, J. Math. **4** (1974), 497–504.

[10] T. Kato, *Perturbation Theory for Linear Operators.* Second edition. Springer-Verlag, Berlin-New York, 1976.
[11] P. D. Lax, *Functional Analysis*, Wiley-Interscience, New York, 2002.
[12] G. I. Taylor, *Diffusion by conitnuous movements*, Proc. London. Math. Soc. **20** (1922), 196–212.

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